Rutgers University: Algebra Written Qualifying Exam

August 2017: Problem 1 Solution

Exercise. Let G be a group and let $H \subset G$ be a proper subgroup containing all other proper subgroup of G. Show the following:

(a) H is normal.

Solution.

H is normal if $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$

If g is contained in some proper subgroup of G, then g^{-1} is also contained in that proper subgroup.

$\implies g, g^{-1} \in H$

$\implies \forall h \in H$, $ghg^{-1} \in H$ since H is closed under multiplication

Suppose g is not contained in any proper subgroup of G.

For any $g \in G$, $\langle g \rangle \subseteq G$ is a subgroup of G.

Since g is not contained in any proper subgroup of G, it follows that $G = \langle g \rangle$.

$\implies G$ is the cyclic group generated by g

Since G is cyclic, G is abelian

$\implies \forall g \in G$ and $h \in H \subset G$,

$$ghg^{-1} = gg^{-1}h = h \in H$$

Thus, H is a normal subgroup of G

(b) G is a cyclic group.

Solution.

(see part a)

(c) G is a finite group.

Solution.

Suppose G is infinite.

If $G = \langle g \rangle$ is infinite, they $\langle g^k \rangle$ create distinct subgroups for all $k \in \mathbb{N}$.

$\implies \langle g^3 \rangle$ and $\langle g^2 \rangle$ are distinct proper subgroups of $G = \langle g \rangle$.

$\implies g^{-2} \in \langle g^2 \rangle \subseteq H$ and $g^3 \in \langle g^3 \rangle \subseteq H$

$\implies g^{-2}g^3 = g \in H$.

But then $H = \langle g \rangle = G$ which is contradiction since H is a proper subgroup.

Thus, we conclude that G must be finite.